A vendor managed inventory model using continuous approximations for route length estimates and Markov chain modeling for cost estimates

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Abstract
We consider a vendor that supplies a set of geographically dispersed retailers and that can monitor the inventory levels at the retailers. Such an arrangement is often called Vendor Managed Inventory (VMI). The dispatch of a vehicle is made to a fixed group of retailers. Normally, the inventory levels at the vendor’s warehouse and at the retailers are established by modeling the problem as a Joint Replenishment Problem (JRP). This fixed set-up cost ignores the differences in distances, number of retailers visited and vehicle load that may occur, in particular when these retailers are served on joint delivery trips.

This paper develops a more detailed specification of the transport costs than JRP models. In order to ensure that the complexity of the problem does not become overwhelming, we assume that the retailers are identical and uniformly distributed across an area, which can either be a two-dimensional area or a one-dimensional line structure (corresponding to e.g. a major traffic artery). The expected travel distances across a given number of retailers can now be estimated analytically, using results from the field of continuous approximation for two-dimensional areas, or using our own approximation for one-dimensional ones. We choose to use a Markov chain approach to minimize transport and inventory model simultaneously. When the routes through the retailers are not fixed, such an approach would require a large number of states if we keep track of all the inventory positions of the retailers. Using our analytic approximations, we are able to obtain more accurate set-up costs than regular JRP approaches, most clearly when demand is infrequent. In addition, the detailed specification of our transport costs means that we can easily derive interesting properties of the resulting distribution set-up, such as the resulting average CO₂ emissions.

Keywords: Joint replenishment policies, Continuous approximation, Markov chain modeling.

1. Introduction

Vendor Managed Inventory (VMI) refers to the situation in which a vendor monitors the inventory level at its retailers and decides when and how much inventory to replenish at each of his retailers. In this way the vendor better utilizes his production equipment as well as provides cost and service efficient inventory control for his retailers. In this paper, we add the factor of geography, in the form of the locations of the retailers, into the analysis of VMI systems.
We consider the replenishment policy as follows. In a joint delivery, a vehicle is dispatched for a delivery tour or route to a group of retailers. There are reasons for having fixed routes along groups of retailers: it is easy to plan fixed routes, and if demand occurs frequently, it is likely that each retailer on a tour needs replenishment. However, if demand is infrequent, only a fraction of the retailers may have faced demand and need replenishment when a vehicle is dispatched. It may then make sense to only visit these retailers on a delivery tour. The routes and the resulting transportation costs are then flexible, which poses large computational challenges. Due to the challenging nature of the problem, we develop an approach for a single item distribution system in this paper. A direction of future research is the extension to the multi-product setting based on our results.

In inventory studies, the problem of having joint deliveries to a set of retailers is known as a Joint Replenishment Problem (JRP). One stream of research assumes deterministic demands, cf Viswanathan (1996) and Wildeman et al (1997). Another stream assumes that demands follow a stochastic process. As our paper accounts for stochastic demand, we focus accordingly on the latter stream of research in this review of JRP studies. The focus in this type of research lies on analyzing good control policies, such as the can order policy, analyzed first by Balinfy (1964) and subsequently by Silver (1981), Federgruen, Groenevelt and Tijms (1984), Schultz and Johansen (1999), Melchiors (2002) and Johansen and Melchiors (2003). The main problem with the can order policy is to develop a valid mathematical model. Others have proposed policies where the coordination of replenishment decisions is secured by only being allowed to replenish at certain time points. This is accomplished by either having deterministic review intervals or a common stochastic review interval governed by the total demand since the last replenishment opportunity. Base-stock policies based on deterministic review intervals are developed by Atkins and Iyogun (1988). If one instead of a deterministic review interval let it be dependent on the total demand since the last replenishment opportunity, then one gets the QS policy proposed by Renberg and Planche (1967) and subsequently analyzed by Pantumsinchai (1992). Viswanathan (1997) formulated a P(s,S) policy, where there is a common deterministic review interval and the replenishment decision of each item is governed by an (s,S) policy. He also proposed a generalization of the P(s,S) policy, which he called the Q(s,S) policy, where the review interval is stochastic as described above. For this policy, Nielsen and Larsen (2005) and later Larsen (2009), for a further extension to the case of correlated demands, have developed a mathematical model and an algorithm to compute optimal policy variables.

In JRP studies, there is a fixed set-up for each delivery, meaning that each delivery to a fixed group of retailers, or zone, costs about the same. The implicit assumption is that the same retailers are visited during each delivery and the cost of an average delivery tour is easily predictable. However, in particular if demand is infrequent, the set-up costs may vary with the number of retailers visited and the transportation distance. Therefore, we study in more detail how they affect the transportation costs arising from the geographical locations of retailers, where it may make sense to adjust the retailers’ inventory strategies to reduce transportation distances, even though this is not a strictly optimal pure inventory strategy.

Where JRP approaches tend to ignore routing considerations, the field of Inventory Routing Problems (IRPs) specifically aims to simultaneously minimize routing and inventory costs. Actually, there is a wide range of IRP approaches; see the overview papers by Baita et al. (1998) and Andersson et al. (2010). Some studies allow for routing flexibility, where, given a range of possible demand distributions, one should decide a time period to visit each retailer. However,
approaches that have flexible routes are usually only able to solve relatively small instances, i.e., with few retailers and time periods, due to the combined complexity of inventory and routing. Dynamic programming approaches for this problem have been presented in Kleywegt et al. (2004) and Hvattum et al. (2009). In order to reduce the complexity of IRPs, many studies set a common inventory policy for fixed groups of retailers. Generally, retailers are replenished with a vehicle that travels along a fixed route. Such policies are collectively known as fixed-partition policies; see e.g. Anily and Federgruen (1990) and Viswanathan et al. (1997). However, these policies do not allow for flexible routing through zones and generally require deterministic demand.

In this paper, we develop JRP inventory policies that address the randomness of demand and the flexibility of routing over a very long time horizon. For inventory modeling over an infinite horizon, a Markov modeling approach is generally applied. A straightforward way of incorporating geography explicitly is to apply a routing approach that determines a shortest route through all retailers needing replenishment in a given period, or alternatively, an IRP approach that also has the option to delay a delivery to a retailer or move it forward. The resulting Markov chain representation, however, could contain a huge number of states, as it needs to keep track of the inventory position of each individual retailer. The complexity is compounded by the route computations in every state. Instead, we assume that retailers are identical and keep track of the inventory position of a single typical retailer. The probability can then be derived that a certain number of retailers \( m \) needs a refill without needing to specify which retailers need replenishment and where they are located.

To be able to estimate the expected distance through \( m \) retailers, we make the assumption that demand is uniformly distributed across the service area of the vendor. This means that the demand rates of the individual retailers are similar and that the locations of retailers are evenly spread across the service area.

We consider two different types of service areas: a two-dimensional circular area (the circular city; actually, the shape is not so relevant, but for the sake of convenience, we use this shape) and a one-dimensional area in which the retailers are assumed to be distributed on a line (the linear city). In both cases, we assume that the distribution of retailers over the area is more or less uniform. These areas are illustrated in Figure 1. Usually, a service area is modeled two-dimensionally, but we choose to add the linear city case, as it can be observed in geographies where cities and towns are located along a major traffic artery. Our motivating example is the eastern Jutland area in Denmark, where cities are located along the E45 motorway, as illustrated by the light distribution (a proxy for population density) in Figure 2. Transport corridors frequently have such a structure as well; see Rodrigue et al. (2004, p. 84)

For the circular city case, we can use the field of continuous approximation (CA) to estimate the lengths of the delivery tours, and thus avoid to perform laborious routing computations. The field of CA is discussed extensively in Daganzo (2004). In general, CA approaches are used for distribution network design questions. In an EOQ-type setting, the transportation costs form part of the ‘set-up costs’ of a delivery. It has been established experimentally and analytically that the distance through an area is a function of the number of retailers and the size of the area (and thus of
the retailer density of the area). For our purposes, the CA approach from Burns et al. (1985) is suitable, as it allows for a variable number of $m$ retailers to be visited on a delivery tour. For the one-dimensional linear city, we derive a similar distance estimate for the travel distance in Section 4.

Using the Markov chain approach, we can analyze various interesting aspects of the distribution system. We assess the influence of geography and compare the circular and linear city structures. Another interesting aspect is the influence of different transportation and inventory components on the optimal inventory policy and zoning strategy. Finally, we evaluate the degree of vehicle utilization and the level of CO$_2$ emissions for different policies.

In Section 2, we describe the type of distribution systems for which our model is valid. Then, Section 3 describes the case of deterministic demand and the optimal inventory decisions in this case. In Section 4, we derive a Markov model for computing the combined inventory and transportation costs in the case of Poisson distributed demand. Section 5 contains computational experiments and Section 6 the conclusion and directions for future research.

2. The distribution system

We consider a distribution system where a single warehouse serves $N$ retailers that are located uniformly distributed in an area surrounding the warehouse. As indicated in the previous section, we consider two structures: a single dimensional line where the warehouse is located in the endpoint (a center position could have been chosen, but for symmetry reasons we consider only one of the halves) or a two-dimensional circular structure where the warehouse is located in the center; see Figure 1. The number $N$ could be rather large, so that it can be too laborious to keep track of specific retailer locations, as argued in the previous section. If demand is uniformly distributed across a specific area, we can consider a typical retailer in a zone instead of all retailers individually, which hugely simplifies the analysis. However, this also implies that we have to make sacrifices in any specific distinction of the retailers, meaning they are identical w.r.t. to cost structure and the demand pattern they face. The retailers face random demand for a particular product. Each retailer faces demand generated by independent Poisson processes all having intensity $\lambda$ arrivals per hour. The arrivals take place only in the open hours so there is no arrival of demands in the closing hours. We assume that the length of the open period is $T_{Op}$ hours ($T_{Op} < 24$), say from 8 am to 6 pm, that is, in this case $T_{Op} = 10$ hours. Correspondingly the number of hours closed $T_{Ci}$ is $T_{Ci} = 24 - T_{Op}$. These opening hours can easily be modified in the model, as long as the delivery to each zone can take place within the closing hours.

The warehouse (or depot) which serves the retailers with this particular product is assumed to have ample supply, so any considerations about inventory control at the supplier are ignored. The focus is on devising a distribution policy that minimizes the back-order and inventory costs at the retailers plus the distribution costs by transporting the goods from the supplier to the retailers. We assume that an information system is in place such that all registered demand at the retailers is immediately transmitted to the supplier. The supplier has a fleet of $M$ containers (or bins), each having the capacity to carry $W$ units of the product. The supplier has accordingly subdivided the total region into $M$ zones, such that any pair of containers is assigned to a specific zone, where the total number of retailers in zone $j$ ($j=1,\ldots,M$) is denoted by $n_j$, where $\sum_{j=1}^{M} n_j = N$. 
The supplier only dispatches a truck (with a container) to a zone, indexed by \( j \), when the total recorded unfilled demand from the zone is at or above level \( V_j \), where \( V_j \leq W \). Therefore, less than filled truck loads can be dispatched. Furthermore, he only dispatches a truck in the night hours. Therefore, the retailers in a particular zone are exposed to the same delivery time risk; thus, their actual geographical position in the zone has no relevance when the retailer designs his control parameter (as seen below, a base-stock policy) for his inventory control policy.

More concretely, this operation is organized in the following way. The supplier has organized \( M \) containers at his place. Each time he monitors a demand through his information system for a retailer at a particular zone, then he takes a unit from his inventory, puts a tag on it with the name and address of the retailer and places it in the container for that zone. If the container at the end of a workday is “filled”, meaning that it contains at least \( V_j \) items and at most \( W \) items, it is attached to a truck ready for dispatch the following night. The truck then completes a delivery tour to all retailers with at least one item in the bin. As the container can be filled in the middle of the day time (and logically if filled during the middle of the day it must contain \( W \) items), as soon as the container is closed and attached to the truck, any orders received in the remaining part of the day, will continue to be tagged but will be placed at a separate track as a visible backlog.

We assume that the transport costs have four cost components: a fixed dispatch component \( \gamma_0 \), a component \( \gamma_1 \) denoting the cost per distance-unit traversed, a component \( \gamma_2 \) denoting the cost per retailer visited, and a cost \( \gamma_3 \) per hour of transport of a unit. The last component reflects that there are load dependent transport costs. This is similar to the transportation cost structure in Burns et al. (1985). It is assumed that the warehouse has “enough” trucks available. However, the number of trucks can very well be below \( M \) as not all zones are served every night.

As noted previously we do not consider explicitly any inventory control policy at the warehouse. However, we do (partly) assess inventory costs at the warehouse because we assume an inventory cost \( h_W \), measured per hour per unit, is incurred for any item sitting in a bin or a backlog rack at the warehouse. One could justify this because the warehouse is using a base-stock control policy such that as soon as an item is placed in the bin, another item is procured to the warehouse. Also, at the retailers, there is an inventory cost \( h_R \), measured per hour per unit. Since some value-adding might take place, it is assumed that \( h_W \leq h_R \).

The aforementioned cost component \( \gamma_3 \) could be interpreted as an inventory cost incurred while an item is sitting on the truck. Each item passes through three stages: sitting in a bin at the warehouse, being in transit on route to a retailer, and sitting on the inventory at a retailer. At each stage, the item incurs inventory costs. In our opinion, it seems reasonable to assume that \( \gamma_3 \geq h_R \). It means that if the travelled distance is the same, it is optimal to unload as early as possible. For example, in the linear city case with the depot located in coordinate 0 and three retailers located in coordinates 1, 2 and 3, respectively, it is most optimal to unload at the retailers 1, 2, and 3 on the way out to retailer 3, rather than first going fully loaded to retailer 3 and unloading at the return trip. We believe this behavior is also observed in practice, justifying our assumption that \( \gamma_3 \geq h_R \).

We assume that all retailers are identical also in terms of the costs implications and customer behavior. Moreover, we assume that all customers of the retailers are patient such that they are backlogged if their demand cannot be filled immediately. In addition to the inventory cost \( h_R \), discussed above, there is a fixed cost \( p_1 \) per unit backlogged as well as a cost \( p_2 \) per unit per hour for having an item on the backlog list.
In a standard JRP model there are retailer specific order costs and joint replenishment costs. The former is \( \gamma_2 \), while the latter is the remaining total transport cost (comprised of the components \( \gamma_0 \), \( \gamma_1 \), and \( \gamma_3 \)). So contrary to the standard JRP models where the joint order cost is a given parameter, the joint order cost in our paper is far more complex to specify, as the derivations in Section 4 illustrate. If the retailer specific order cost is considerable, a good inventory control policy should have included a re-order point, such that a retailer is served only if his inventory position (inventory position denotes the net-inventory plus the demand registered at the supplier but not yet delivered and net-inventory denotes the on-hand inventory minus the backlog) is below this level. In our paper, we assume that the magnitude of \( \gamma_2 \) is rather small, such that it is natural to limit ourselves to the assumption that retailers have a single policy parameter namely an order-up-to level; a similar structure for the standard JRP can be seen in Pantumsinchai (1988). Furthermore, because all retailers in a zone are exposed to the same delivery risk, they should have the same target for this constant inventory position.

To summarize, given a zoning policy \( N^* = (n_1, \ldots, n_M) \), we consider each zone \( j \) separately. Each retailer has an order up to level or inventory position \( S_j \), and as soon as the open-bin at the supplier reaches a level \( V_j \) a dispatch is made to Zone \( j \) the following night (containing at most \( W \) items) so we denote the policy for Zone \( j \) by \( (V_j, S_j) \). In the process, the vendor incurs the inventory costs at the warehouse (in the bin), in the vehicle and at the retailer, and the transportation costs. In the next section, we specify the long-term cost of having the policy \( \Pi = (n_j, V_j, S_j; j=1,\ldots,M) \). We focus on finding an optimal policy \( (S_j, V_j) \) within a zone, given \( N \) and \( M \) and the geographic structure, where we assume that a division into \( M \) zones exists, including how many retailers \( (n_j) \). By determining \( V_j \) and \( S_j \) for various values of \( M \) and \( n_j \), we can find an effective zoning structure. So our main objective is to derive an optimal policy within a zone, which then enables us to compare different zoning strategies.

### 3. Delivery and zoning

In this section, we analyze the zoning strategies for the distribution system described in the previous section. The distribution strategy consists of the division of the service area into zones that are served jointly and of the determination of the inventory and routing decisions for the individual zones.

The choice for our control policy \( (S_j, V_j) \) for a given zone \( j \) excludes some actions: If a zoning strategy has been chosen, it is not possible to construct a delivery tour that visits retailers in multiple zones. Moreover, it is also not possible to skip the delivery to a retailer far away from the others in a delivery tour through a zone. This is mainly for analytic purposes; if one were to include flexible routing, a dynamic IRP approach such as the one from Hvattum et al. (2011) should be applied. Such approaches can determine (near-)optimal schedules consisting of both a schedule indicating when each retailer is visited and the delivery routes, but they can only cover a limited number of retailers, demand scenarios and time periods due to the complexity of analyzing joint inventory and routing decisions.

The purpose of zoning is to construct groups of retailers that can be served jointly as cost effectively as possible. A set of retailers can be a zone because of the retailers’ demand characteristics. For example, if one group of retailers needs replenishment every second day and
another group every fourth day, it could make sense to serve these two groups of retailers on separate delivery tours. This consideration plays a key part in fixed partition policies; see e.g. Anily and Federgruen (1992). However, since retailer demands are assumed to be stochastic, independent and identically distributed in our case, we cannot take advantage of demand characteristics. We thus decide to construct zones on a geographical basis, meaning that zones consist of closely located retailers.

In case of the circular city, the ideal zoning strategy consists of pie-shaped zones (Burns et al. 1985), i.e., the depot should be in the center and the zones should radiate outward from the center. Even if the area is not circle-shaped, similar zoning strategies still turn out to be effective (Daganzo et al. 2004). This also implies that zones in the circular city case are almost identical: two zones of, say, 10 retailers have the same size.

The number of zones follows from the transportation costs relative to the inventory costs. It is found in among others Burns et al. (1985) that, if possible, full vehicles should be dispatched (that is, for deterministic demand). The amount delivered to each retailer is equal to the demand until the next delivery. If zones are large, daily demand within the zone is large as well and deliveries are frequent. Inventories at the retailers and the warehouse are relatively small, but transportation distances through the zones can be quite long. This set-up is most effective when inventory costs are prominent. On the other hand, when transportation costs are large, it pays off to have smaller zones, each of them with smaller transportation distances, but also with infrequent deliveries.

To the best of our knowledge, the linear city case has not been considered in any of these problems. We propose that the entire area should be divided into intervals \([a_j,b_j]\), where the zones \(j\) are ordered such that \(a_{j+1}=b_j\). The zones are symmetrically distributed on either side of the location 0 of the depot; if the number of zones is odd, one zone will have the depot in its middle. The following example illustrates why zones should be formed by non-overlapping intervals.

**Example:**
Consider a linear city case with six retailers located at \(1,2,\ldots,6\) with the depot at 0. Assume that we construct two zones on that side of the depot, with zone 1 containing retailers located at 1, 2, and 4 and zone 2 the retailers at 3, 5, and 6. We assume that all three retailers in both zones are visited on a delivery tour. Each delivery tour through zone 1 has length \(2\times4=8\), and each delivery tour through zone 2 has length \(2\times6=12\). If the retailer at 3 is transferred to zone 1 and the one at 4 to zone 2, we have that the tour lengths are \(2\times3=6\) and \(2\times6=12\), respectively, with \(a_1=0.5\), and \(b_1=a_2=3.5\). The distance reduction is achieved by the zones closest to the depot. Note that this result requires that the demand occurrences at retailers occur independently; if the retailer at 4 faces demand at the same days as 1 and 2, then a zone containing 1, 2 and 4 can well be cost-effective due to savings in inventory costs at the retailers.

Each zone \(j\) on one side of the depot has a different distance to the depot, denoted by \(a_j\), implying that the zones are no longer identical. We assume that the vehicle delivers to the first retailer encountered, then continues the tour delivering to all the retailers, before returning empty from the last retailer.

Also in the linear city case, smaller zones may be preferred if transportation costs per kilometer are high. In the example above, we compare the zoning strategy with one zone covering all retailers on the side of the zone, and one with two zones \((1,2,3 \text{ and } 4,5,6)\). With full vehicles and deterministic
demand, the total number of dispatches in both strategies is the same, say, one per two days. However, in the one-zone strategy, each delivery tour passes by all six retailers, whereas in the two-zone strategy, each retailer is visited each fourth day. As a consequence, inventories in the two-zone strategy are twice as large as in the one-zone strategy. Transportation distances are smaller, as there is a transportation haul to retailer 3 and back every fourth day instead of a delivery all the way to retailer 6.

4. Total cost computations

We will now show how, for a given policy \( \Pi \), to determine the expected total cost incurred per day (transport, inventory and backorder costs) when observing the system over an infinite time horizon. Denote this term by \( C_{Tot}^{LR}(\Pi) \).

It can be decomposed into

\[
C_{Tot}^{LR}(\Pi) = \sum_{j=1}^{M} \left[ C_{Transp-Zonej}^{LR}(\Pi) + n_j C_{Inv-RetZonej}^{LR}(\Pi) \right]
\]

(1)

Here \( C_{Transp-Zonej}^{LR}(\Pi) \) are the expected daily inventory costs at the warehouse as well as the gross transport cost for dispatching items to zone \( j \), when observing the system in the long run and using policy \( \Pi \). The term gross transport cost covers, in addition to transport costs, also expected in-transit inventory costs and inventory costs for the retailers for housing any incoming shipments until opening the next day. The term \( C_{Inv-RetZonej}^{LR}(\Pi) \) denotes the expected daily inventory and backorder costs for any retailer in zone \( j \) in the long run when using policy \( \Pi \). Both terms \( C_{Transp-Zonej}^{LR}(\Pi) \) and \( C_{Inv-RetZonej}^{LR}(\Pi) \) are computed using a Markov chain methodology. As we analyze a given zone \( j \) and its policy \((n_j, V_j, S_j)\) in isolation, we will omit the index \( j \) in the remainder of this section to ease the notation.

Our Markov chain analyses focus on a single state variable \( v \), which is the content of the backlog at the start of a work-day. In order to have a finite state space we assume that if the backlog at any time exceeds a number \( BL_{max} \), all units in excess of this critical number are sent to the retailer during the night hours using an alternative transport option at a unit cost \( p_0 \). When this cost parameter is large, the model avoids this option. Furthermore, if the input data are chosen in a reasonable way, the system should have sufficient capacity (because only one container can be dispatched per night) such that expediting in reality never takes place. So this is mainly a modeling maneuver to have a finite state space. In the following the parameter \( BL_{max} \) is set such that \( BL_{max} \geq W \). The evolution of the state variable is given by the transition matrix \( Q_{v,v'} \) which denotes the probability for, at the start of the next workday, to have a backlog of size \( v' \) given at the start of the previous workday it was \( v \). Define by \( D_{Zone-ToT}^{Zone-ToT} \) the total demand of all retailers in zone \( j \) the considered day. This random variable is Poisson distributed with mean \( \lambda n_j T_{Op} \). Then \( Q_{v,v'} \) looks as follows for \( v = 0, \ldots, BL_{max} \):
\[
Q_{v,v'} = \begin{cases} 
    P \left( V - v \leq D_{\text{Tot}}^{\text{Zone}} \leq W - v \right) + P \left( D_{\text{Tot}}^{\text{Zone}} = -v \right) & v' = 0 \\
    P \left( D_{\text{Tot}}^{\text{Zone}} = W - v + v' \right) & v' = 1, \ldots, v - 1 < V - 1 \\
    P \left( D_{\text{Tot}}^{\text{Zone}} = v' - v \right) + P \left( D_{\text{Tot}}^{\text{Zone}} = W + v' - v \right) & v' = v, \ldots, V - 1 \\
    P \left( D_{\text{Tot}}^{\text{Zone}} \geq W + BL_{\text{max}} - v \right) & v' = V_{\text{max}} 
\end{cases}
\]

Note that the second part in the first term (case \( v' = 0 \)) only gives a positive value in case \( v = 0 \). Some parts in this specification vanish depending on the value of \( v \). If \( v \geq V \) there will for sure be a nightly dispatch. The first part in the third term is redundant and we can therefore rewrite (2) to

\[
Q_{v,v'} = \begin{cases} 
    P \left( V - v \leq D_{\text{Tot}}^{\text{Zone}} \leq W - v \right) + P \left( D_{\text{Tot}}^{\text{Zone}} = -v \right) & v' = 0 \\
    P \left( D_{\text{Tot}}^{\text{Zone}} = W - v + v' \right) & v' = \max(v - W, 0) + 1, \ldots, V_{\text{max}} - 1 \\
    P \left( D_{\text{Tot}}^{\text{Zone}} \geq W + BL_{\text{max}} - v \right) & v' = V_{\text{max}} 
\end{cases}
\]

It is possible to compute the equilibrium probabilities, denoted by \( P(BL=v) \). We do this by the successive overrelaxation method; see Tijms (2003; pp 108-109). Furthermore, we use, for a new value of \( V \), the previously computed equilibrium probabilities as initial input to the method in order to increase the computation speed. It may seem strange that we base our state description on information about the backlog at the start of a work day. If one pursued a Markov decision model where an optimal policy is found for each state, then it would of course be more natural to express the state based on information at the end of the workday just before the departure time of any truck. However, we use a Markov chain where we evaluate the economic consequences of a given policy (recall that for each zone, the policy is of type \((S, V)\)), so this policy structure is superimposed on our Markov chain. By varying the policy parameters we find the optimal policy; it does not really matter at which time of the day we describe the state of the system. Therefore, we just choose the most convenient time: the start of a workday. Needless to say, better policies might be found by the Markov decision approach. However, the structure of such an optimal policy might not be easily implementable because it does not necessarily have a neat structure. This is why we have decided to follow the Markov chain approach.

### 4.1 Transportation, central warehouse and in-transit inventory costs

In this subsection we assess the term \( C_{\text{Transport-Zone}}^{LR}(II) \). It consists of transport costs and inventory costs at the warehouse. In addition it also includes the inventory costs the retailers incur on the nightly dispatched items and any costs of expediting. In the following we specify these components in detail. When dispatching a container with a content of \( u \) items (where \( u=V, V+1, \ldots, W \)), we must assess the probability that this dispatch is destined for \( m \) retailers (where \( m = 1,2,\ldots,u \)), and denote this probability \( Pr_{\text{Visit}}(u,m) \). It can be derived recursively by incrementing \( u \) by 1, starting with \( u=1 \). Clearly \( Pr_{\text{Visit}}(1,1) = 1 \). When defining \( Pr_{\text{Visit}}(u,m) = 0 \) if \( m=0, m>n, \) or \( m > u \). It can be specified as
\[ PrVisit(u, m) = PrVisit(u - 1, m) \frac{m}{n} + PrVisit(u - 1, m - 1) \frac{n - m + 1}{n} \]  
(4)

Concerning the expected number of retailers visited when dispatching a bin of size \( u \), it holds (see also Burns et al, 1985), that

\[ \sum_{m=1}^{\infty} mPrVisit(u, m) = n \left(1 - \left(\frac{n-1}{n}\right)^u\right) \]  
(5)

Denote by \( E[L(m)] \) the expected travelled distance through \( m \) retailers in the zone. It will be specified further in the following. If the content of the container destined for the zone is \( v \) units at the beginning of the work day, the expected transport cost, though without the load dependent costs (specified by component \( \gamma_3 \)), for the possible dispatch the following night can be estimated as:

\[ y_0P \left( D_{Top}^{Zone-Tot} \geq V - v \right) + \]
\[ \sum_{u=0}^{W-v-1} P \left( D_{Top}^{Zone-Tot} = u \right) \left( y_2 n \left(1 - \left(\frac{n-1}{n}\right)^{u+v}\right) + y_1 \sum_{m=1}^{u+v} Prvisit(u + v, m) E[L[m]] \right) + \]
\[ P \left( W - v \leq D_{Top}^{Zone-Tot} \right) \left( y_1 \sum_{m=1}^{W} PrVisit(W, m) E[L[m]] \right) + \]
\[ y_2 n \left(1 - \left(\frac{n-1}{n}\right)^W\right) + p_0 E \left[ \max \left( D_{Top}^{Zone-Tot} - W - BL_{max} + v, 0 \right) \right] \]  
(6)

This expression contains, firstly, that with probability \( P \left( D_{Top}^{Zone-Tot} \leq V - v - 1 \right) \) (which may be zero if \( v \geq V \), no dispatch is made the following night and accordingly no transport costs are incurred. The first term in (6) concerns the expected fixed dispatch cost (component \( \gamma_0 \)), the second term the expected cost wrt. distance (component \( \gamma_1 \)) and number of visits (component \( \gamma_2 \)) of dispatching a not completely filled bin, while the third term contains the similar expected costs when enough demand from the zone has accumulated to dispatch a completely filled bin at the end of the day. Finally the fourth term denotes the expected expediting costs. A further specification of \( E[L(m)] \) in (6) depends on the spatial structure. In the one-dimensional line structure, the retailers in the zone are assumed to be uniformly located in an interval \([a, b]\), as depicted in Figure 1. The length of the route is specified by the retailer farthest away. This location is specified by \( \max\{Z_k: k=1,\ldots,m\} \) where all \( Z_k \) are independent and uniformly distributed on the interval. The distribution function of \( \max\{Z_k: k=1,\ldots,m\} \) is \( F(z) = \left(\frac{z-a}{b-a}\right)^m, a < z < b \) and the density function is \( f(z) = \frac{m(z-a)^{m-1}}{(b-a)^m} \). Therefore the travel distance of a dispatch to this zone that must visit \( m \) retailers is

\[ E[L(m)] = \int_a^{b} (z-a)f(z)dz = 2 \left[ a + \int_a^{b} (z-a)f(z)dz \right] = 2 \left[ a + \frac{m}{m+1} (b-a) \right] \]  
(7)

In the two-dimensional structure, the distance through \( m \) retailers can be computed using the CA approach from Burns et al (1985). However, the approach generally requires \( m \) to be sufficiently large to have a fixed distance \( D' \) to and from the zone which is fixed and independent of \( m \). In order to take into account that a zone can contain only a small number of retailers, we make the following modification, as described in Turkensteen and Klose (2012). \( E[L(m)] \) can be specified as

\[ E[L(m)] = 2E(r)/m + K \frac{m-1}{m} \sqrt{\frac{mN}{\rho}} \]  
(8)
The first term is the expected distance to or from the zone (the headway length) and consists of the term $E(r)$, denoting the expected distance from any point in the area to the central location. This term should be divided by the number of retailers in the zone $m$ to obtain the distance to the two closest locations from the depot (Daganzo, 2004). In a circular area with radius $R$, it can be computed that the expected distance $E(r)$ from the center to any retailer is $\frac{2}{3}R$, giving a headway length from and to the zone of $\frac{4}{3m}R$. The second term in (8) is an approximation of the distance (Traveling Salesman like) traversed in the zone, the detour length. The constant $K$ is normally put to 0.6 and the term $\rho$ is the density of the retailers (the density is the same for all zones). Usually, the term $\frac{m-1}{m}$ is omitted, but we choose to maintain it in order to take small zones into account as well.

The expected daily inventory costs at the warehouse when there are $v$ items in the open-bin at the start of the work-day, is:

$$h_w \left[ T_{op} \left( v + \frac{1}{2} \lambda n T_{op} \right) + T_{ct} \left\{ \sum_{u=0}^{V-v-1} P \left( D_{T_{op}}^{zone-Tot} = u \right) (u + v) \right. \right.$$

$$\left. + \sum_{u=W-v+1}^{\infty} P \left( D_{T_{op}}^{zone-Tot} = u \right) \min(u + v - W, BL_{max}) \right] \right]$$

(9)

The first term is the expected inventory during the day time (starting at level $v$ and piling up with rate $\lambda n_j$ per hour), see also Tijms (2003; Example 2.2.2), the second term is the expected inventory costs incurred during the night hours when the bin is not dispatched (note here that when the standard convention for notation of a summation is followed, the summation vanishes if the upper summation limit is smaller than the lower summation limit, that is, $v > V$) while the third term is the expected inventory costs incurred in the night hours of those items that could not be in the container dispatched to the zone, excluding those expedited.

Finally, we assess the inventory costs during transportation. These costs are most relevant when having a unit in the vehicle is expensive, for example, due to cooling. In order not to be caught in too many details, we assume that a transport haul departs from the depot exactly as the night period starts and it returns to the depot exactly as the new work-day starts, and the speed of the transport is adjusted accordingly. We agree that this is indeed a simplification; the underlying assumption is that we keep open at what time the vehicle departs from the depot and arrives at the retailers. If the policy is to deliver as late as possible, i.e., with the last delivery finishing at $T_{op}$, the computation can be adjusted. However, as can be seen in (18), our specification leads to a very reasonable specification of the inventory costs that should be charged a unit in transit over the night.

In the analysis, we interpret the cost component $\gamma_3$, the cost per hour of transport of a unit, as an inventory cost and call it the in-transit inventory cost. During the transport towards the zone, all items on the bin are in transit; after that, there are stops in which part of the bin is offloaded to a retailer and then subject to an inventory cost $h_R$ per hour until the opening time) after which the remaining items are transported to the other retailers. Finally, the vehicle makes a haul back to the depot empty, where no additional inventory or load-dependent transportation costs are incurred. Below, we compute the expected in-transit inventory costs of a unit on a delivery tour through $m$ retailers.
We decompose the term $E[L(m)]$ into

$$E[L(m)] = E[LF(m)] + E[LU(m)] + E[LE(m)] \quad (10)$$

These are respectively the expected distance traversed of the truck being full on the headway, unloading in the detour and being empty on the headway back. The stop time at each retailer is $f$ hours, where $f < T_{Cl}/n$. As noted above we assume the speed of the truck is timed such that the tour takes exactly $T_{Cl}$. This means the speed $s$ is given as

$$s = \frac{E[L(m)]}{T_{cl} - mf} \quad (11)$$

Assuming the bin unloads in a linear rate, we can then assess the overnight inventory costs of a unit sitting in the truck that visits $m$ customers as

$$C_{UnitINVON}(m) = \frac{T_{cl} - fm}{E[L(m)]} \left( \gamma_3 \left( E[LF(m)] + \frac{1}{2} (E[LU(m)]) \right) + h_R \left( \frac{1}{2} E[LU(m)] + E[LE(m)] \right) \right) + \frac{1}{2} (h_R + \gamma_3) fm \quad (12)$$

A further specification of three components in the decomposition in (9) depends on the spatial structure. For the one-dimensional structure of Figure 1, we have

$$E[LF(m)] = a + \frac{b-a}{m+1} \quad (13)$$

$$E[LU(m)] = \frac{m-1}{m+1} (b - a) \quad (14)$$

and

$$E[LE(m)] = a + \frac{m}{m+1} (b - a) \quad (15)$$

For the two-dimensional structures, the specifications are as follows:

$$E[LF(m)] = E[LE(m)] = \frac{2}{3m} R \quad (16)$$

$$E[LU(m)] = \frac{m-1}{m} K \sqrt{\frac{mN}{p}} \quad (17)$$

In this two-dimensional case (11) can be considerably simplified to

$$C_{UnitINVON}(m) = \frac{1}{2} T_{Cl} (\gamma_3 + h_R) \quad (18)$$

Thus in a two-dimensional and symmetric structure, information about stop times is redundant. Also for the one-dimensional case one can simplify (12) through (13)-(15) to
\[ C_{\text{UnitInv}}(m) = \frac{T_{\text{Cl}}-f_{m}}{E[L(m)]} \left( \gamma_3 \left( \frac{b+a}{2} + h_R \left( a + (b-a) \left( \frac{1.5m-0.5}{m+1} \right) \right) \right) + \frac{1}{2} (h_R + \gamma_3) f_{m} \right) \]  

(19)

For both the one and two-dimensional case, the expected in-transit inventory cost of a dispatched bin that has a backlog \( v \) at the start of the day can now be specified in a similar way to (6) as

\[
\sum_{u=V-v}^{U=v-1} P(D_{\text{Tot}}^{\text{Zone}}=u)(u+v) \sum_{m=1}^{m+v} \text{PrVisit}(u+v,m) C_{\text{UnitInv}}(m)
\]

\[
+ P(W-v \leq D_{\text{Tot}}^{\text{Zone}}) W \sum_{m=1}^{W} \text{PrVisit}(W,m) C_{\text{UnitInv}}(m)
\]

(20)

Let the term \( C_{\text{Trans}}(v) \) denote the expected daily inventory cost at the warehouse plus the expected gross transport cost of any dispatched vehicle the following night when starting the day with a backlog \( v \). Thus \( C_{\text{Trans}}(v) \) is the summation of (6), (9) and (20). Weighting this with the steady state probabilities \( P(BL=v), v=0,\ldots,V_{\text{max}} \) gives us

\[
C_{\text{Transp-Zone}}(\Pi) = \sum_{v=0}^{V_{\text{max}}} C_{\text{Trans}}(v) P(BL=v)
\]

(21)

It should be underlined that it is not an exact measure for the transport cost that is derived through (21), as the estimates of the travel distances and the overnight inventory costs rest on approximations. However, as the simulation studies in the next section reveal, the error from these approximations is fairly limited.

4.2 Retailer inventory holding and stock-out costs

We will now derive the term \( C_{\text{Inv-RetZone}}^{LR}(\Pi) \). To that end, we need to decompose the steady state equilibrium distribution for the backlog, destined for the zone, into an equilibrium probability distribution for the net-inventory of any randomly chosen retailer in the zone. Let the random variable \( NI \) denote the net-inventory at a retailer in Zone \( j \) at the start of a workday. Its steady state probability distribution can be derived in the following manner. If the net inventory is \( i \) then the backlog destined for zone \( j \) at the warehouse must at least be \( v \geq S - i \). For any value of \( v \) the probability that \( S-i \) items in this backlog are destined for the particular retailer is

\[
\left( \frac{V}{S-i} \right) \left( \frac{1}{n} \right) \left( \frac{n-1}{n} \right)^{S-i-1} v^{-S+i}
\]

as all retailers in the zone are identical. Therefore we reach a similar expression as in Pantumsinchai (1992), namely:

\[
P(NI = i) = \sum_{v=S-i}^{B_{\text{max}}} \left( \frac{V}{S-i} \right) \left( \frac{1}{n} \right) \left( \frac{n-1}{n} \right)^{S-i-1} v^{-S+i} P(BL = v)
\]

(22)

Denote by \( C_{\text{Inv}}(i,t) \) the expected accumulated inventory and backorder cost that a retailer incurs during a workday until time \( t \), when starting the day with net inventory \( i \). By conditioning on the first arrival (which is a random variable \( U \) that is exponentially distributed with mean \( 1/\lambda \) and density function \( f(u) = \lambda e^{-\lambda u}, u>0 \)), the expected inventory incurred during the whole workday can be specified as:
Here $I_A$ is a so-called indicator function taking the value of 1 if the condition $A$ is true and otherwise 0. Following Rosling (2002), (22) can more conveniently be rewritten in the following way. Namely, focus on when $u$ hours have evolved during the workday and then by $D_u$ denote the random variable that measures the retailer’s total demand recorded during that day until time $u$. Due to the assumption of a Poisson process $D_u$ is Poisson distributed with mean $\lambda u$. The expected on-hand inventory at time $u$ is then

$$C_{\text{Inv}}(i, T_{Op}) = \int_0^{T_{Op}} u(h_R \cdot \max\{i, 0\} + p_2 \cdot \max\{-i, 0\}) + p_1 \cdot I_{(i \leq 0)}$$

$$+ C_{\text{Inv}}(i - 1, T_{Op} - u) f(u) \, ds + P(U \geq T_{Op}) T_{Op} (h_R \cdot \max\{i, 0\} + p_2 \cdot \max\{-i, 0\})$$

(23)

Finally, we must assess the expected inventory and backorder cost during the night hours (here excluding those in the possible incoming shipment as these have been accounted for already). It can be specified as

$$C_{\text{Inv}}(i, T_{Op}) = \int_0^{T_{Op}} u(h_R \cdot \max\{i, 0\} + p_2 \cdot \max\{-i, 0\}) + p_1 \cdot I_{(i \leq 0)}$$

$$+ C_{\text{Inv}}(i - 1, T_{Op} - u) f(u) \, ds + P(U \geq T_{Op}) T_{Op} (h_R \cdot \max\{i, 0\} + p_2 \cdot \max\{-i, 0\})$$

(23)

$$O H(i, u) = \begin{cases} 
0 & i \leq 0 \\
\sum_{y=0}^{\infty} (i - y) P(D_u = y) & i > 0 
\end{cases}$$

(24)

While the expected backlog at time $u$ is

$$B L(i, u) = \begin{cases} 
-i + \lambda u & i \leq 0 \\
\sum_{y=0}^{\infty} (y - i) P(D_u = y) & i > 0 
\end{cases}$$

(25)

During the work-day the expected increment in the backlog is $BL(i, T_{Op}) - BL(i, 0)$. Therefore we obtain

$$C_{\text{Inv}}(i, T_{Op}) = h_R \int_0^{T_{Op}} O H(i, u) du + p_2 \int_0^{T_{Op}} B L(i, u) du + p_1 (B L(i, T_{Op}) - B L(i, 0))$$

(26)

This term can be further rewritten to an expression only containing finite sums. When $i > 0$,

$$C_{\text{Inv}}(i, T_{Op}) = p_2 T_{Op} \left(-i + \frac{1}{2} \lambda T_{Op}\right) + \frac{h_R + p_2}{\lambda} \sum_{y=0}^{i-1} (i - y) \left[1 - e^{-\lambda T_{Op}} \sum_{x=0}^{y} \frac{(\lambda T_{Op})^x}{x!}\right]$$

$$+ p_1 \left[\lambda T_{Op} - i + e^{-\lambda T_{Op}} \sum_{x=0}^{i-1} (i - x) \frac{(\lambda T_{Op})^x}{x!}\right]$$

(27)

and when $i \leq 0$,

$$C_{\text{Inv}}(i, T_{Op}) = p_2 T_{Op} \left(-i + \frac{1}{2} \lambda T_{Op}\right) + p_1 \lambda T_{Op}$$

(28)

Finally, we must assess the expected inventory and backorder cost during the night hours (here excluding those in the possible incoming shipment as these have been accounted for already). It can be specified as

$$C_{\text{RetInvON}}(i) = \begin{cases} 
T_{Cl}(p_2 (-i + \lambda T_{Op}) + (h_R + p_2) \sum_{y=0}^{i-1} (i - y) e^{-\lambda T_{Op}} \frac{(\lambda T_{Op})^y}{y!}) & i > 0 \\
T_{Cl} p_2 (-i + \lambda T_{Op}) & i \leq 0 
\end{cases}$$

(29)
Note here that in this specification any backorder can first be cleared at the start of the next workday. Given an inventory level \( i \) at the start of the day, we can specify the expected inventory and backorder cost of a retailer during a full day as

\[
C_{\text{Retinv}}(i) = C_{\text{inv}}(i, T_{\text{op}}) + C_{\text{retinv0n}}(i)
\]  

(30)

In order to find the expected daily inventory and backorder cost in the long run, we then need to weight with the steady state probability distribution \( P(NI= i), i=S-V_{\text{max}}, \ldots, S \). That is,

\[
C_{\text{inv-RetZone}}^{LR}(\Pi) = \sum_{i=S-V_{\text{max}}}^{S} C_{\text{Retinv}}(i) P(NI = i)
\]  

(31)

We have now outlined how to compute \( C_{\text{inv-RetZone}}^{LR}(\Pi) \) and \( C_{\text{inv-RetZone}}^{LR}(\Pi) \) and can thereby evaluate the cost performance of policy \( \Pi \) by (1). By systematically varying the policy parameters, one can find the minimum cost policy \( (S_j, V_j) \) for a given zone \( j \); this is done for each zone. Through variation of the zoning set-up, one can then derive which zoning strategy makes sense for the case at hand.

### 5. Numerical examples

In this section we conduct various numerical investigations. These are organized around the following topics: accuracy of the model’s performance compared to simulations, optimal number of zones in the linear and circular city, the dependence of \( (S, V) \) on the zoning structure, comparisons to a standard JRP model with less detail on the transportation cost structure. Finally, we illustrate how our model can contribute to the ongoing discussion about “sustainability” by evaluating the impact of a zoning and a \( (S, V) \) policy on the expected CO\(_2\) emissions.

#### Accuracy of the model’s performance compared to simulations

In order to test the accuracy of the model from Section 4, we have developed a simulation model for the linear city case, programmed in Arena; see Kelton et al (2007). In the table below we have first used our model to find the optimal solution for the given dataset. Thereafter we have collected data for the given optimal policy parameters on its cost performance through simulation.

<Table 1 about here>

The numbers between parentheses in the cost column present the half-length of the 95% confidence interval. We see that the simulated results follow closely that of our computational Markov chain model, though we see that the simulation model in general predicts slightly higher costs. We believe that this happens because our Markov model, possibly via Jensen’s inequality, tends to underestimate the distance dependent shipment costs. As also commented in Section 4, an exact estimate on some of the transport related cost elements is hard to make. The last column contains another output from the simulation model, namely the average vehicle load. This emphasizes that the policy variable \( V \) is the minimum dispatched quantity and the actual dispatched quantity is often larger.

#### Optimal number of zones
Here we set the parameters $N$, $\lambda$, $W$ such that $W$ is slightly above $\lambda N T_{op}$. This means that in principle we could have a single zone and then on average make a dispatch every night. The reason why it must be “slightly” above is due to randomness in the demands. If we set $W$ “strictly equal to” it, there would be a build-up of expedited items in the long run, in a similar vein as that one should set the utilization below the number of servers in a standard queuing model. Specifically, we set $N = 50$, $\lambda = 0.1$ and $W = 60$. The other parameters are set as in Table 1. For the linear city case we have varied the number of zones from 1 to 5 in Table 2 and collected the minimum total cost. In addition we have collected the part of the costs attributed to the cost parameters $\gamma_1$ and $\gamma_3$, which we vary in the experiments.

As can be seen, the optimal number of zones for this case is 3, but the cost differences are rather small. Moreover, we see that the impact from $\gamma_1$ and $\gamma_3$ decreases substantially when going from 1 zone to 5 zones. If one argues that there might be some hidden environmental costs that are not adequately covered by these two cost components, the results here indicate that from an environmental perspective the zoning strategy is good. In order to further study this, we do the same experiment but we halve $\gamma_1$ and $\gamma_3$ to 250 and 0.75, respectively, in Table 3 and double $\gamma_1$ and $\gamma_3$ to 1000 and 3, respectively, in Table 4.

In Table 3, the optimal number of zones is 2; the higher transportation costs due to longer distances are compensated by the lower inventory costs due to more frequent deliveries to retailers.

In Table 4 the optimal number of zones is shifted from 3 to 4 and the no-zoning strategy is very inoptimal. Compared to Table 2, of course, the impact of the transportation costs is increased. This example illustrates how the VMI set-up changes as a consequence of increases in well-understood cost drivers, such as mileage and ton kilometers, which in turn could be caused by an increased emphasis on transport costs or awareness that environmental costs should be accounted for.

We have done a similar analysis for the circular city, where we have chosen to use a radius $R = 7$, where the costs are similar to the linear city case above. Using the same other data as in Table 2 and varying the number of zones from 1 to 9 we obtain the results in Table 5.

Now the optimal number of zones, 8, in the circular city case is clearly larger than in the linear city case. We believe the reason is the square root expression in (8). In both the linear and circular city cases, a decrease in the number of zones generally leads to lower inventory costs, as each retailer is visited relatively frequently and as fewer bins are needed at the central warehouse. However, if we merge two zones in the circular city case, the detour length increases by a factor $\sqrt{2}$ (assuming full truck loads), whereas the headway length usually decreases by much less. Moreover, the load dependent costs of some retailers can increase, as the load is transported over a longer distance to this specific retailer. In the linear city case, the transportation costs are highest for the zones furthest away from the depot. However, if two zones far away from the depot are merged, the delivery tour
length is mainly determined by the headway towards and from the zones, which hardly changes by merging the zones. Therefore, the minimum cost solution to the circular city case usually contains more zones than the minimum cost solution to the linear city case with the same cost parameters.

**How does \((S,V)\) depend on the zoning structure**

We only consider the one-dimensional structure here. A good example of how \((S,V)\) depends on the zone can be seen in Table 1. The zone closest to the warehouse has the lowest \(V\). As the distance from the zone to the warehouse increases, the transportation costs per item increase and it becomes optimal to consolidate shipments to a higher degree, i.e., to increase \(V\). As transport hauls are made less frequently, retailers increase their base stock levels. Therefore, for the linear city with two zones \(j\) and \(j+1\) where zone \(j\) is closest to the depot we obtain the relations: \(S_j \leq S_{j+1}\) and \(V_j \leq V_{j+1}\).

**Comparisons to a standard JRP model with less detail on the transportation cost structure**

Here, we compare our approach to a standard JRP model. Recall that such a model does not allow for flexible route lengths and number of stops. As our model uses a single policy variable for each retailer, namely the order-up-to level, the most natural standard JRP model to compare our model to would be the one from Pantumsinchai (1992), with a joint order cost per zone and with no further specification of its decomposition. In that approach, we should compute a gross \(\gamma_0\) component for each zone and then put all the other \(\gamma\) components equal to zero.

We consider two different distribution situations. In the first case, \(W=60\) and the number of retailers per zone is about 12 or 13. Thus, if vehicles are fully loaded, there are on average about 5 items for each retailer, meaning that it is likely that all retailers are visited. In order to create variations in \(m\), we adjusted the data with an increase of \(N\) to 500 and a decrease of \(\lambda\) to 0.01; aggregate demand is the same as before. This dataset corresponds to situations with very low-frequent demand, which for instance characterizes a “spare part” while we might coin the former “a commodity”.

First, we analyze the linear city case. For the commodity case, we take the data from Table 4 with 4 zones and fix the optimal \((S,V)\) solutions. In the spare part case, we take 4 zones with 125 retailers in each zone. In the following we report on this experiment where we use two different methods for assessing the gross \(\gamma_0\) components. In order to clarify we take our point of departure in the data of Table 4 with the optimal number of 4 zones. With the optimal \((S,V)\) solutions locked we identify the transport costs (the impact from the \(\gamma\) cost parameters) and delivery frequencies to each zone. The gross \(\gamma_0\) components could then be found by dividing the expected daily transport costs with the delivery frequency. The result of this can be seen in the column “Gross \(\gamma_0\) (model)” in Table 6a. This approach may seem odd as it requires that we first use our model with the detailed transport cost specification. Another approach is therefore in a more rough way to calculate a gross \(\gamma_0\) component. We assume that the truck always departs full and that it visits all retailers in the zone. Take for example Zone 3. Here one could derive the gross \(\gamma_0\) component as \((2*(12/13)*2.6 + 4.8)*1000 + 7*60*3 + 10*13 + 50 = 15840\). In a similar way the other gross \(\gamma_0\) components, reported in the last column of Table 6a, are computed.

<Table 6a about here>
Using the gross $\gamma_0$ components as inputs, we determine optimal $(S,V)$ policies. Our results are stated in Table 6b, which contains the gross $\gamma_0$ components calibrated from our model and the gross $\gamma_0$ components computed in the rougher manner (by hand).

A potential problem with fixed gross $\gamma_0$ components is that there is a tendency to set $V$ too high, as the dependence of the delivery cost on the vehicle load is not taken into account. The delivery costs and hence the total costs are then underestimated. This can be observed in Table 6b. However, the cost and the optimal policy parameters are almost the same for our approach and the standard JRP approaches. A reason could also be that, on average, almost all retailers in the zone are visited in each dispatch. Moreover, since $V$ is close to $W$, full vehicles are generally dispatched.

In the spare part case, we expect different results, with dispatch to only a fraction of the retailers and possibly lower vehicle loads. By replicating the experiment we obtain:

The total minimum daily cost is now 20173.47. Compared to the previous value of 15163.39 (see Table 4) this is a considerable increase, which is caused by a diminishing risk pooling effect by increasing $N$ and decreasing $\lambda$. As one often observes with low-frequent spare parts, the optimal value of $S$ is 1 for all zones. When transforming to a standard JRP model, as described before, we obtain the results as reported in Table 7b.

Although the standard JRP model obtains accurate results for the spare part case as well, the cost estimates and the values of $V$ are higher when the hand calculated gross $\gamma_0$ values are used. As could be expected, the gross $\gamma_0$ component computed by hand is inaccurate, as only a fraction of the retailers in a zone is visited and this strongly overestimates the delivery costs. This is most pronounced for Zone 1, where trucks probably dispatch with less than full truck loads as the optimal policy parameter $V$ is 49. The problem is then that the value of $\gamma_0$ cannot be determined from fixed quantities such as $W$ or $n$, but that the value needs to be computed in a more detailed way, which ultimately leads to our approach.

A similar analysis is performed on the circular city, with data from Table 5, but in order to have a feasible 4 zone policy, we set $R=3$. Now the routing can be quite complex and it may make sense to fix the routes for this reason. The results in Table 8a show that due to the relatively high transportation costs in our case, the vehicle is well-filled and all retailers are likely to be visited during a dispatch for the commodity case. For the spare part case, the standard JRP approaches overestimate $V$. Because $V$ is close to $W$ in our approach, the differences in costs are small, but if transportation costs are lower or inventory costs are higher, the difference becomes significant. For example, we find that for the spare part case with $\gamma_1=100$, our approach finds the optimal policy $(S,V) = (1,30)$, whereas the optimal policy determined with a model-based gross $\gamma_0$ component is $(S,V) = (1,43)$. 


In the circular city case, it may make sense to use fixed routes. Such a strategy is easy to handle administratively, as it is not necessary to compute new route and schedules for every delivery. The set-up costs are the same for every delivery tour, with the exception of the in-transit inventory costs. However, in case of infrequent demand, it may be that the fixed routes pass by many retailers without demand, leading to unnecessary transportation costs. Table 8b illustrates that for commodity products, the increase in distribution costs of fixed routes can be negligible and probably outweighs the costs of having flexible routes. As demand rates decrease, however, flexible routing becomes increasingly sensible. Our approach can thus be used as a tool to determine whether flexible or fixed routing should be preferred.

An additional advantage of our modeling approach is that it enables us to derive interesting characteristics of the resulting distribution characteristics, because it allows us to track in detail how transportation is performed instead of imposing a gross cost factor to it. We demonstrate this in the following subsection.

**Impact of policies: vehicle utilization and carbon emissions**

The detailed specification of delivery costs, based on transportation distances, stops, and the transported allows us to measure the impact of policy decisions on several characteristics of the distribution, such as the degree of vehicle utilization and the amount of CO$_2$ emissions.

More specifically, for a given policy $(S, V)$ one can determine the delivery frequency as the proportion of days with a delivery, the average distance per day, and the average number of stops per day. The term related to $\gamma_3$ is the number of hours that items spend in a vehicle, or item hours.

The emissions of carbon dioxide (CO$_2$) and other greenhouse gases, the carbon emissions, are major contributors to global warming and transportation is one of the factors with an increasing impact (McKinnon et al., 2010). Many organizations are becoming increasingly environmentally aware for many different reasons: their consumers and other stakeholders require it, it can give a competitive advantage, or legislation is introduced, such as carbon prices, road tariffs, and fuel charges. An integral part of this is carbon accounting, where an organization is able to measure how much greenhouse gas is emitted; see also McKinnon et al. (2010). According to Bektas and Laporte (2011), carbon emission levels are proportional to the energy and fuel use, for a given type of vehicle. We illustrate how our approach enables the measurement of CO$_2$ emissions (and energy use) as a consequence of the choice of a distribution policy i.e., a combination of zoning and inventory policies $(S,V)$.

The paper by Bektas and Laporte (2011) relates energy use and thus CO$_2$ emissions to various explanatory factors, such as distance, travel speed and vehicle load. We assume that the characteristics of our distribution system are as in Table 1 from Bektas and Laporte (2011), so speed is 40 km per hour everywhere, the weight of an empty vehicle is 3, and its maximum load is 4 units. Based on these results, we perform the calculations reported in Appendix A and find that an empty vehicle uses 0.2478 l diesel per km and the transportation of a full load of $W$ units requires 0.1754 l diesel. The carbon emissions of a liter of diesel are 2.32, so the emissions per km of the vehicle itself are 0.3267 kg and of the full load inside the vehicle 0.2132 kg; for $W=30$, this amounts to emissions per item per km of 0.0077 kg.
In the model from Bektas and Laporte (2011) as we adapt it, carbon emissions are partly proportional to the average daily distance (distance-based), and partly to the load of the vehicle (load-based). It does not directly account for emissions caused by having a dispatch (the vehicle should pick up the items) and for having a stop (the vehicle slows down and accelerates), but they can be calculated easily from the delivery frequency and the average number of stops per day. Distance based emissions are directly proportional to the average distance per day; load-based emissions are more difficult to compute, as our model does not track them directly.

The determination of load-based CO$_2$ emissions in the linear city case is shown with the following example. Take zone 1 from Table 9a, situated at the interval (0;2], where the 10 retailers demand 10 items per day and the average distance to a retailer is 1. So the average number of item kilometers, which we define as the sum of the number of kilometers traveled by all items per day, equals 10. Since the emission factor per item kilometer is 0.0077, total emissions are $0.0077 \times 10 = 0.08$. The direct computation is valid for the linear city case because each item always travels $x$ km to a retailer located at $x$ km from the depot. That explains why the total load-based CO$_2$ emissions are 1.93 for all distribution set-ups in Table 9a to c.

For the circular city, the approach above is invalid, as the route of an item from the depot to its destination retailer depends on the number of retailers that are also visited on the same route, which in turn depends on the choice of the policy $(S,V$ and the zoning strategy). Instead, one could approximate the number of item kilometers from the number of item hours (related to $\gamma_3$ in our approach) by dividing the item hours by the average duration of a delivery tour (14 hours in our model) and multiplying the resulting number by the average distance covered in a delivery tour. For example, for zone 1 in the solution from Table 8b, the average number of item hours per day is 39.09. The average distance per tour is 3.62, computed as the distance per day (1.27) divided by the delivery frequency (0.35). The number of item kilometers is therefore $\frac{39.09}{14} \times 3.62 = 2.79 \times 3.62 = 10.11$, a slight overestimation of the real amount of 10, and the daily load-based emissions are 0.08.

A strategy for reducing carbon emissions is the dispatch of well utilized vehicles. We measure also the average vehicle utilization at the start of a delivery tour, which is, in the long term, simply the average demand at zone $j$, $\lambda_n T_{Op}$, divided by the zone’s delivery frequency.

We analyze the standard linear city case, where we consider an increase of the cost component $\gamma_1$, in our case from a start value of $\gamma_1=100$ to $\gamma_1=500$. A possible reason could be a change in tariff structure from a new Danish government initiative, imposed on the transport where the tariff is based more on kilometers driven and less on a fixed tariff for having the vehicle (Jyllands-Posten, 2012). Of course, the real impact of such tariffs would be well below the cost increase as we investigate it, but we wish to illustrate how the change in solution affects CO$_2$ emissions and vehicle utilization. For $\gamma_1=100$, the minimum cost solution contains 3 zones in which the optimal policies $(S,V)$ are (3,19), (3,22) and (3,24) for zones 1, 2, and 3, respectively. In addition we consider the more costly 5 zone solution for $\gamma_1=100$, with the optimal policies (3,14), (4,22), (4,23), (4,24), and (4,25) for the zones 1 to 5, respectively, in order to separate the effects of the zoning and of the inventory policy $(S,V)$. The distribution characteristics are reported in Tables 9a and 9b.

<Tables 9a and 9b about here>
When increasing $\gamma_1$ to 500 the results are given in Table 9c, where the optimal policies are $(4,25)$, $(4,27)$, $(5,29)$, $(5,29)$ and $(5,29)$ for the zones 1 to 5, respectively.

For our instance, roughly 80% of the emissions are related to distance only, whereas the remaining 20% relates to load as well. The results indicate that an increase in $\gamma_1$ has a dual effect on CO$_2$ emissions. Firstly, it becomes optimal to dispatch vehicles less frequently, leading to better vehicle utilization and a decrease in distance-based CO$_2$ emissions (and energy use). Secondly, the number of zones is increased as the relative inventory costs have decreased. Though this decreases the degree of vehicle utilization, we observe an additional decrease in distance-based CO$_2$ emissions due to the fact that zones have become smaller.

This example illustrates how our detailed specification of transportation costs can be used to compute and compare carbon emission levels. An interesting direction of future research is to look more deeply into the relationships between geography, inventory policies, zoning, and CO$_2$ emissions for different types of vehicles and products and for different vehicle velocities.

6. Conclusions and future research

In this paper, we have studied a Joint Replenishment Problem (JRP) from a central depot to geographically dispersed retailers. The set-up costs of a delivery depend on the locations of the retailers that are visited in that delivery tour; if not all retailers are visited, these costs may vary. However, current JRP approaches use a fixed set-up cost for each delivery. Inventory routing approaches do allow for flexible routing, but cannot take into account a large number of retailers, a long time horizon and random demand simultaneously.

We choose to reduce the routing complexity by assuming that retailer locations are uniformly distributed in an area. The routing costs can now be computed using results from the field of continuous approximation (CA); for the case that retailers are located on a line (e.g. a major traffic artery), we have derived an alternative distance estimate. The resulting cost model takes into account in which distribution stage an item is at any time: at the retailers, the warehouse or in transit. We have developed a Markov chain modeling approach for determining optimal dispatch policies and optimal inventory policies at the retailers.

Our approach is most suitable in cases where demand is relatively infrequent. In that case, there is a large amount of uncertainty about the timing of the delivery, the quantity to be delivered and the number of retailers to be visited. Moreover, the detailed specification of the transportation costs allows us to measure how the choice for an inventory policy influences the resulting vehicle utilization, driving distances and the expected CO$_2$ emissions from transportation. This is important if a company wishes to assess the environmental impact of its distribution.

Our approach assumes a distribution system with a single centrally located warehouse, identical and geographically uniformly distributed retailers, and a base-stock inventory control policy. An interesting future research direction is to relax the limitations imposed on the distribution system, for example by having vehicles dispatched throughout the day but possibly with a delay. A related research direction is the following: It is assumed that the warehouse has ample supply and puts an item in a zone’s bin when an item is purchased at a retailer in that zone. However, the items are
only needed when a dispatch is made; any items put in the bin earlier are sitting there idle. So it
could be useful to derive optimal inventory strategies at the warehouse. Finally, the applicability
of the model can be increased by taking the distribution of multiple products into account that are
delivered jointly to retailers.

Acknowledgement
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entitled Management design and evaluation of sustainable freight and logistics systems.

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Appendix A: CO\textsubscript{2} emission calculations

Here, we explain how the CO\textsubscript{2} emission factors for the empty vehicles and the emissions per load unit have been obtained. In this appendix, we use the computations and the numerical example from Bektas and Laporte (2011), where the example is from their Table 1. We consider the same type of vehicle and speed, i.e., the load of the vehicle weighs 4 weight units and the empty vehicle 3, and speed is constant at 40 km/h, the area is flat and a medium-heavy vehicle is used. As in the example, we choose to omit such factors as acceleration, declination of the road, and vehicle type.

Consider an arbitrary connection from \( i \) to \( j \) with distance \( d_{ij} \) and average velocity \( v_{ij} \). The parameter \( \alpha_{ij} \) is an arc specific constant, the term \( f_{ij} \) denotes the load from \( i \) to \( j \) and \( \beta \) is a vehicle specific constant. The energy use on the connection from \( i \) to \( j \), \( P_{ij} \), in KWh is then:

\[
P_{ij} = \alpha_{ij}(w + f_{ij})d_{ij} + \beta v_{ij}^2 d_{ij}
\]  

(A1)

We find that the used parameters are \( \alpha_{ij} = 0.0435 \) for every arc \((i,j)\), and \( \beta = 0.000073 \). The energy use for an empty vehicle \( P_{ij}^{\text{empty}} \) from \( i \) to \( j \), related only to the distance traveled, is then:

\[
P_{ij}^{\text{empty}} = (\alpha_{ij} w + \beta v_{ij}^2) d_{ij}
\]  

(A2)

The energy use related to the vehicle load \( P_{ij}^{\text{load}} \) depends on the load times distance \( f_{ij} d_{ij} \), e.g., tonne kilometers, as follows:

\[
P_{ij}^{\text{load}} = \alpha_{ij} f_{ij} d_{ij}
\]  

(A3)

Since \( \alpha_{ij} \) and \( v_{ij} \) are constant everywhere, so \( \alpha_{ij} = \alpha = 0.0435 \) and \( v_{ij} = v = 40 km/h \). We find that the energy use per km for an empty vehicle is 0.2478 \((=\alpha w + \beta v^2)\) and for the full load of vehicle per km, it is 0.1754 \((=4 \alpha\), where 4 is the full load of the vehicle\).

The CO\textsubscript{2} emissions are in this model proportional to the fuel use. One liter of fuel generates 8.8 KWh, of which 20\% is effectively used. The emissions per liter of diesel are thus 2.32 kg of CO\textsubscript{2}. So the CO\textsubscript{2} emissions per effective KWh are 2.32/(0.2\times8.8) = 1.3182. Emissions per km traveled are thus 0.3267 for the empty vehicle and 0.2312 for transporting the full load in the vehicle.
Figures and Tables

Figure 1: Examples of the circular city (left) and linear city (right) cases

Figure 2: Map showing the density of people that can reach a point within a 15 minutes drive (the darker the color, the more people) with the E45 motorway added. Source: http://www.sm.dk/data/Dokumentertilpublikationer/IM/Regionalpolitisk_redeg%C3%B8relse2001/bil03.htm
<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
<th>Simulated output</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zone</td>
<td>$n$</td>
<td>$a$</td>
</tr>
<tr>
<td>1</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>10</td>
<td>8</td>
</tr>
</tbody>
</table>

Table 1: Linear city, $N=50$, $\lambda = 0.1$, $W = 30$, $L=10$, $T_{Op} = 10$, $T_{Cl} = 14$, $h_R = 0.5$, $h_W = 0.4$, $\gamma_0 = 50$, $\gamma_1 = 500$, $\gamma_2 = 10$, $\gamma_3 = 1.5$, $p_0 = 1000$, $p_1 = p_2 = 5$. $M = 5$ zones and of equal size. The simulations (in all 20 replications) are done over 1000 days starting with all retailers having full stocks at level $S$ and no orders at the warehouse.

<table>
<thead>
<tr>
<th>No of zones</th>
<th>Minimum total costs</th>
<th>Impact from $\gamma_1$ and $\gamma_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10918.53</td>
<td>8420.33 (77.12%)</td>
</tr>
<tr>
<td>2</td>
<td>9445.87</td>
<td>6508.00 (68.90%)</td>
</tr>
<tr>
<td>3</td>
<td>9356.69</td>
<td>5887.68 (62.92%)</td>
</tr>
<tr>
<td>4</td>
<td>9598.88</td>
<td>5663.14 (59.00%)</td>
</tr>
<tr>
<td>5</td>
<td>9966.98</td>
<td>5600.38 (56.19%)</td>
</tr>
</tbody>
</table>

Table 2: Linear city, $N=50$, $\lambda = 0.1$, $W = 60$, $L=10$, $T_{Op} = 10$, $T_{Cl} = 14$, $h_R = 0.5$, $h_W = 0.4$, $\gamma_0 = 50$, $\gamma_1 = 500$, $\gamma_2 = 10$, $\gamma_3 = 1.5$, $p_0 = 1000$, $p_1 = p_2 = 5$. Each zone contains the same number of retailers (best possible) if possible; otherwise, there are slightly more in the outer zones (e.g. for the case with 3 zones, there are 16 retailers in the nearest zone and 17 in the others).

<table>
<thead>
<tr>
<th>No of zones</th>
<th>Minimum total costs</th>
<th>Impact from $\gamma_1$ and $\gamma_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>6677.08</td>
<td>4521.64 (67.70%)</td>
</tr>
<tr>
<td>2</td>
<td>6135.85</td>
<td>3404.39 (55.48%)</td>
</tr>
<tr>
<td>3</td>
<td>6344.99</td>
<td>3137.00 (49.44%)</td>
</tr>
<tr>
<td>4</td>
<td>6677.64</td>
<td>3100.84 (46.44%)</td>
</tr>
<tr>
<td>5</td>
<td>7066.93</td>
<td>3130.77 (44.30%)</td>
</tr>
</tbody>
</table>

Table 3: Linear city, $N=50$, $\lambda = 0.1$, $W = 60$, $T_{Op} = 10$, $T_{Cl} = 14$, $h_R = 0.5$, $h_W = 0.4$, $\gamma_0 = 50$, $\gamma_1 = 250$, $\gamma_2 = 10$, $\gamma_3 = 0.75$, $p_0 = 1000$, $p_1 = p_2 = 5$. Each zone contains the same number of retailers (best possible) if possible; otherwise, there are slightly more in the outer zones, see comments in Table 2.
<table>
<thead>
<tr>
<th>No of zones</th>
<th>Minimum total costs</th>
<th>Impact from $\gamma_1$ and $\gamma_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>19327.59</td>
<td>16788.84 (86.86%)</td>
</tr>
<tr>
<td>2</td>
<td>15909.09</td>
<td>12883.18 (80.98%)</td>
</tr>
<tr>
<td>3</td>
<td>15199.08</td>
<td>11630.76 (76.52%)</td>
</tr>
<tr>
<td><strong>4</strong></td>
<td><strong>15163.39</strong></td>
<td><strong>11041.26 (72.82%)</strong></td>
</tr>
<tr>
<td>5</td>
<td>15388.54</td>
<td>10713.58 (69.62%)</td>
</tr>
</tbody>
</table>

**Table 4:** Linear city, $N=50$, $\lambda = 0.1$, $W = 60$, $L = 10$, $T_{Op} = 10$, $T_{Cl} = 14$, $h_R = 0.5$, $h_W = 0.4$, $\gamma_0 = 50$, $\gamma_1 = 1000$, $\gamma_2 = 10$, $\gamma_3 = 3$, $p_0 = 1000$, $p_1 = p_2 = 5$. The number of retailers in each zone (best possible) is equal but with slightly more in the outer zones, see comments in Table 2.

<table>
<thead>
<tr>
<th>No of zones</th>
<th>Minimum total costs</th>
<th>Impact from $\gamma_1$ and $\gamma_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>23966.89</td>
<td>21518.27 (89.78%)</td>
</tr>
<tr>
<td>2</td>
<td>20267.39</td>
<td>17289.28 (85.31%)</td>
</tr>
<tr>
<td>3</td>
<td>18102.50</td>
<td>14542.69 (80.34%)</td>
</tr>
<tr>
<td>4</td>
<td>16827.20</td>
<td>12667.19 (75.28%)</td>
</tr>
<tr>
<td>5</td>
<td>16067.33</td>
<td>11313.22 (70.41%)</td>
</tr>
<tr>
<td>6</td>
<td>15680.54</td>
<td>10366.63 (66.11%)</td>
</tr>
<tr>
<td>7</td>
<td>15472.68</td>
<td>9607.87 (62.10%)</td>
</tr>
<tr>
<td>8</td>
<td><strong>15436.73</strong></td>
<td><strong>8986.42 (58.21%)</strong></td>
</tr>
<tr>
<td>9</td>
<td>15501.76</td>
<td>8477.56 (54.69%)</td>
</tr>
</tbody>
</table>

**Table 5:** Circular city, $N=50$, $\lambda = 0.1$, $W = 60$, $R = 7$, $T_{Op} = 10$, $T_{Cl} = 14$, $h_R = 0.5$, $h_W = 0.4$, $\gamma_0 = 50$, $\gamma_1 = 500$, $\gamma_2 = 10$, $\gamma_3 = 1.5$, $p_0 = 1000$, $p_1 = p_2 = 5$. The used zoning strategy is a “cakeform” where the number of retailers in each zone (best possible) is equal.

<table>
<thead>
<tr>
<th>Zone</th>
<th>A</th>
<th>B</th>
<th>Sopt</th>
<th>Vopt</th>
<th>Daily Transp. cost</th>
<th>Delv. freq.</th>
<th>Gross (model) $\gamma_0$</th>
<th>Gross (hand) $\gamma_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>2.4</td>
<td>6</td>
<td>52</td>
<td>1104.05</td>
<td>0.2104</td>
<td>5247.39</td>
<td>5830</td>
</tr>
<tr>
<td>2</td>
<td>2.4</td>
<td>4.8</td>
<td>6</td>
<td>55</td>
<td>2116.11</td>
<td>0.2043</td>
<td>10357.86</td>
<td>10630</td>
</tr>
<tr>
<td>3</td>
<td>4.8</td>
<td>7.4</td>
<td>6</td>
<td>57</td>
<td>3419.69</td>
<td>0.2183</td>
<td>15665.09</td>
<td>15840</td>
</tr>
<tr>
<td>4</td>
<td>7.4</td>
<td>10</td>
<td>6</td>
<td>58</td>
<td>4549.56</td>
<td>0.2175</td>
<td>20917.52</td>
<td>21040</td>
</tr>
</tbody>
</table>

**Table 6a:** Linear city with input data as in Table 4.

<table>
<thead>
<tr>
<th>Zone</th>
<th>Normal $\gamma_0$ (model)</th>
<th>Gross $\gamma_0$ (hand)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Total costs</td>
<td>Sopt Vopt</td>
</tr>
<tr>
<td>1</td>
<td>2048.38</td>
<td>(6,53)</td>
</tr>
<tr>
<td>2</td>
<td>3086.52</td>
<td>(6,56)</td>
</tr>
<tr>
<td>3</td>
<td>4443.68</td>
<td>(6,58)</td>
</tr>
<tr>
<td>4</td>
<td>5584.81</td>
<td>(6,58)</td>
</tr>
<tr>
<td></td>
<td><strong>15163.39</strong></td>
<td><strong>15162.47</strong></td>
</tr>
</tbody>
</table>

**Table 6b:** The standard JRP analysis using data from Table 6a and the gross $\gamma_0$ (model) values and the gross $\gamma_0$ (hand) values.
<table>
<thead>
<tr>
<th>Zone</th>
<th>$a$</th>
<th>$B$</th>
<th>$S_{opt}$</th>
<th>$V_{opt}$</th>
<th>Daily Transp. cost</th>
<th>Delv. freq.</th>
<th>Gross (model) $\gamma_0$</th>
<th>Gross (hand) $\gamma_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>2.5</td>
<td>1</td>
<td>49</td>
<td>1361.39</td>
<td>0.2279</td>
<td>5973.63</td>
<td>7520</td>
</tr>
<tr>
<td>2</td>
<td>2.5</td>
<td>5</td>
<td>1</td>
<td>53</td>
<td>2451.25</td>
<td>0.2164</td>
<td>11327.40</td>
<td>12520</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>7.5</td>
<td>1</td>
<td>55</td>
<td>3497.51</td>
<td>0.2126</td>
<td>16451.13</td>
<td>17520</td>
</tr>
<tr>
<td>4</td>
<td>7.5</td>
<td>10</td>
<td>1</td>
<td>56</td>
<td>4542.37</td>
<td>0.2111</td>
<td>21517.62</td>
<td>22520</td>
</tr>
</tbody>
</table>

Table 7a: Linear city: $N = 500$, $\lambda = 0.01$, $W = 60$, $L = 10$, $T_{Op} = 10$, $T_{Cl} = 14$, $h_R = 0.5$, $h_W = 0.4$, $\gamma_0 = 50$, $\gamma_1 = 1000$, $\gamma_2 = 10$, $\gamma_3 = 3$, $p_0 = 1000$, $p_1 = p_2 = 5$.

<table>
<thead>
<tr>
<th>Zone</th>
<th>Normal Gross $\gamma_0$ (model)</th>
<th>Gross $\gamma_0$ (hand)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Total costs</td>
<td>(Sopt,Vopt) Computed costs</td>
</tr>
<tr>
<td>1</td>
<td>3363.10 (1,50)</td>
<td>3361.57</td>
</tr>
<tr>
<td>2</td>
<td>4521.54 (1,54)</td>
<td>4520.59</td>
</tr>
<tr>
<td>3</td>
<td>5610.39 (1,55)</td>
<td>5610.15</td>
</tr>
<tr>
<td>4</td>
<td>6678.44 (1,56)</td>
<td>6679.49</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Zone with</th>
<th>(S,V)</th>
<th>Cost flexible route</th>
<th>Cost fixed route</th>
</tr>
</thead>
<tbody>
<tr>
<td>Commodity</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>n=13</td>
<td>(6,55)</td>
<td>2565.14</td>
<td>2585.81</td>
</tr>
<tr>
<td>n=12</td>
<td>(6,54)</td>
<td>2356.80</td>
<td>2375.85</td>
</tr>
</tbody>
</table>

Table 8a: Circular city with input data as in Table 5: $W = 60$, $R = 3$, $T_{Op} = 10$, $T_{Cl} = 14$, $h_R = 0.5$, $h_W = 0.4$, $\gamma_0 = 50$, $\gamma_1 = 500$, $\gamma_2 = 10$, $\gamma_3 = 1.5$, $p_0 = 1000$, $p_1 = p_2 = 5$; where $N = 50$, $\lambda = 0.1$, (commodity case) and $\lambda = 0.01$, $N = 500$ (spare part case).
Solution with 3 zones; \( n_1 = 16, n_2 = n_3 = 17; \ W = 30 \)

<table>
<thead>
<tr>
<th>Zone</th>
<th>Overall costs</th>
<th>Delivery frequency</th>
<th>Distance/day</th>
<th>Number of stops / tour</th>
<th>Item hours</th>
<th>Vehicle load</th>
<th>Load</th>
<th>Distance</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1238.54</td>
<td>61.38%</td>
<td>3.64</td>
<td>12.92</td>
<td>60.34</td>
<td>26.07</td>
<td>0.20</td>
<td>1.19</td>
<td>1.39</td>
</tr>
<tr>
<td>2</td>
<td>1755.54</td>
<td>60.72%</td>
<td>7.73</td>
<td>13.84</td>
<td>91.54</td>
<td>28.00</td>
<td>0.64</td>
<td>2.53</td>
<td>3.17</td>
</tr>
<tr>
<td>3</td>
<td>2170.75</td>
<td>58.93%</td>
<td>11.52</td>
<td>14.02</td>
<td>101.06</td>
<td>28.85</td>
<td>1.09</td>
<td>3.76</td>
<td>4.85</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>5164.84</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.93</td>
<td>7.48</td>
<td>9.41</td>
</tr>
</tbody>
</table>

**Table 9a:** Characteristics of the minimum cost solution (3 zones) to the linear city case with \( N = 50, \ \lambda = 0.1, W = 30, L=10, T_{Op} = 10, T_{Cl} = 14, h_R = 0.5, h_W = 0.4, \gamma_0 = 50, \gamma_1 = 100, \gamma_2 = 10, \gamma_3 = 1.5, p_0 = 1000, p_1 = p_2 = 5. \)

Solution with 5 zones, \( n = 10 \) for all zones; \( W = 30 \)

<table>
<thead>
<tr>
<th>Zone</th>
<th>Overall costs</th>
<th>Delivery frequency</th>
<th>Distance/day</th>
<th>Number of stops / tour</th>
<th>Item hours</th>
<th>Vehicle load</th>
<th>Load</th>
<th>Distance</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>748.66</td>
<td>51.80%</td>
<td>1.85</td>
<td>8.60</td>
<td>39.09</td>
<td>19.30</td>
<td>0.08</td>
<td>0.61</td>
<td>0.68</td>
</tr>
<tr>
<td>2</td>
<td>948.14</td>
<td>37.80%</td>
<td>2.88</td>
<td>9.36</td>
<td>55.17</td>
<td>26.46</td>
<td>0.23</td>
<td>0.94</td>
<td>1.17</td>
</tr>
<tr>
<td>3</td>
<td>1102.01</td>
<td>36.72%</td>
<td>4.26</td>
<td>9.41</td>
<td>60.27</td>
<td>27.23</td>
<td>0.39</td>
<td>1.39</td>
<td>1.78</td>
</tr>
<tr>
<td>4</td>
<td>1248.68</td>
<td>35.81%</td>
<td>5.59</td>
<td>9.46</td>
<td>62.76</td>
<td>27.92</td>
<td>0.54</td>
<td>1.83</td>
<td>2.37</td>
</tr>
<tr>
<td>5</td>
<td>1391.72</td>
<td>35.07%</td>
<td>6.88</td>
<td>9.49</td>
<td>64.23</td>
<td>28.51</td>
<td>0.69</td>
<td>2.25</td>
<td>2.94</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>5439.20</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.91</td>
<td>7.01</td>
<td>8.94</td>
</tr>
</tbody>
</table>

**Table 9b:** Characteristics of the 5 zone linear city solution with \( N = 50, \ \lambda = 0.1, W = 30, L=10, T_{Op} = 10, T_{Cl} = 14, h_R = 0.5, h_W = 0.4, \gamma_0 = 50, \gamma_1 = 100, \gamma_2 = 10, \gamma_3 = 1.5, p_0 = 1000, p_1 = p_2 = 5. \)

Solution with 5 zones, \( n = 10 \) for all zones; \( W = 30 \)

<table>
<thead>
<tr>
<th>Zone</th>
<th>Overall costs</th>
<th>Delivery frequency</th>
<th>Distance/day</th>
<th>Number of stops / tour</th>
<th>Item hours</th>
<th>Vehicle load</th>
<th>Load</th>
<th>Distance</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1312.51</td>
<td>35.07%</td>
<td>1.27</td>
<td>9.49</td>
<td>39.09</td>
<td>28.51</td>
<td>0.08</td>
<td>0.41</td>
<td>0.49</td>
</tr>
<tr>
<td>2</td>
<td>2021.23</td>
<td>34.01%</td>
<td>2.59</td>
<td>9.55</td>
<td>55.17</td>
<td>29.40</td>
<td>0.23</td>
<td>0.85</td>
<td>1.08</td>
</tr>
<tr>
<td>3</td>
<td>2703.08</td>
<td>33.44%</td>
<td>3.89</td>
<td>9.57</td>
<td>60.27</td>
<td>29.90</td>
<td>0.39</td>
<td>1.27</td>
<td>1.66</td>
</tr>
<tr>
<td>4</td>
<td>3374.47</td>
<td>33.44%</td>
<td>5.22</td>
<td>9.57</td>
<td>62.74</td>
<td>29.90</td>
<td>0.54</td>
<td>1.71</td>
<td>2.25</td>
</tr>
<tr>
<td>5</td>
<td>4044.26</td>
<td>33.33%</td>
<td>6.54</td>
<td>9.58</td>
<td>64.22</td>
<td>30.00</td>
<td>0.69</td>
<td>2.14</td>
<td>2.83</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>13455.55</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1.93</td>
<td>6.37</td>
<td>8.30</td>
</tr>
</tbody>
</table>

**Table 9c:** Characteristics of the minimum cost solution (5 zones) to the linear city solution with \( N = 50, \ \lambda = 0.1, W = 30, L=10, T_{Op} = 10, T_{Cl} = 14, h_R = 0.5, h_W = 0.4, \gamma_0 = 50, \gamma_1 = 500, \gamma_2 = 10, \gamma_3 = 1.5, p_0 = 1000, p_1 = p_2 = 5. \)